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Ito's theorem and scalar field theory

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Abstract. Starting from the stochastic formulation of scalar field theory, Ito's theorem is used to derive a functional equation for the generating functional. Solving this equation the usual Feynman diagram rules are recovered.

1. Introduction

Recently there has been much interest in the similarities between quantum field theory and the theory of (functional) stochastic differential equations†. The solutions of both these theories are characterised by their Green functions, that is averages of products of the field variables. It is conjectured that by choosing a particular form of stochastic differential equation both theories will produce the same Green function. The proof of this conjecture leads to a new (the so-called stochastic) prescription for quantising fields.

Adopting Ito's formulation of stochastic calculus leads immediately to an equation for the functional which generates the Green functions. For the free field this equation is easily solved in closed form. No closed form solution can be found for the interacting field theory ($\lambda\phi^4$) but perturbation theory leads to the usual Feynman diagram rules.

In § 2 the stochastic quantisation hypothesis is formally stated. In §§ 3 and 4 the equation for the generating functional is derived and solved. The paper is concluded, in § 5, with a discussion of the outlook for stochastic field theory.

2. Stochastic quantisation

Classically the dynamics of a field $\phi(x)$, defined over d spacetime coordinates x, is given in terms of the action $S[\phi(x)]$ by

$$\delta S[\phi(x)]/\delta\phi(x)=0. \tag{2.1}$$

Moving to the quantum level equation (2.1) is replaced by, in Euclidean space, the path integral expression

$$Z_{\mathrm{FT}}=\frac{\int \mathrm{D}[\phi]\exp(-S[\phi(x)]+\int \mathrm{d}x\,\phi(x)J(x))}{\int \mathrm{D}[\phi]\exp(-S[\phi(x)])}, \tag{2.2}$$

† See for example Parisi and Wu (1981) or Floratas and Iliopoulos (1983).

where Z_{FT} is the generating functional for the Green function as can be seen by functionally differentiating with respect to the source $J(x)$.

Now consider the (stochastic) field $\phi(x, \tau)$ defined over spacetime coordinates x and an additional coordinate τ with action $S[\phi(x, \tau)]$ and let the dynamics with respect to the new coordinate be given by the Ito stochastic differential equation†

$$\mathrm{d}\phi(x, \tau) = -\frac{\delta S[\phi(x, \tau)]}{\delta\phi(x, \tau)}\mathrm{d}\tau + \sqrt{2}\,\mathrm{d}W(x), \tag{2.3}$$

(where $W(x)$ is the white noise process), and impose initial conditions

$$\phi(x, 0) = 0 \qquad \forall x. \tag{2.4}$$

It is conjectured that, in the limit of large τ, the ensemble average of products of $\phi(x, \tau)$ field approach the corresponding field theoretic Green functions calculated from equation (2.2).

Introducing the stochastic generating functional

$$Z_{\mathrm{stoc}}(\tau) = \left\langle \exp \int \mathrm{d}x\, \phi(x, \tau) J(x) \right\rangle_{\mathrm{ensemble}}, \tag{2.5}$$

the conjecture becomes

$$\lim_{\tau\to\infty} Z_{\mathrm{stoc}}(\tau) = Z_{\mathrm{FT}}. \tag{2.6}$$

3. Ito's theorem and the generating functional

The essential nature of stochastic calculus is contained in Ito's theorem which states that if a field $\phi(x, \tau)$ obeys equation (2.3) then the functional $F[\phi(x, \tau)]$ obeys

$$\mathrm{d}F[\phi(x, \tau)] = \int \mathrm{d}y\left[\left(-\frac{\delta S}{\delta\phi(y, \tau)}\frac{\delta F}{\delta\phi(y, \tau)} + \frac{\delta^2 F}{\delta\phi(y, \tau)^2}\right)\mathrm{d}\tau + \sqrt{2}\frac{\delta F}{\delta\phi(y, \tau)}\mathrm{d}W(y)\right]. \tag{3.1}$$

In particular choosing $F[\phi(x, \tau)] = \exp(\int \mathrm{d}x\, \phi(x, \tau)J(x))$ and averaging over the ensemble gives

$$\frac{\partial}{\partial\tau}\left\langle \exp\left(\int \mathrm{d}x\, \phi(x, \tau)J(x)\right)\right\rangle_{\mathrm{ensemble}} = \left\langle \int \mathrm{d}y\left(-\frac{\delta S}{\delta\phi(y, \tau)}J(y) + J^2(y)\right)\exp\left(\int \mathrm{d}x\, \phi(x, \tau)J(x)\right)\right\rangle_{\mathrm{ensemble}} \tag{3.2}$$

which can be rewritten using equation (2.5) as‡

$$\frac{\partial Z_{\mathrm{stoc}}}{\partial\tau}(\tau) = \int \mathrm{d}y\left[-J(y)\frac{\delta S}{\delta\phi}\left(\frac{\delta}{\delta J(y)}\right) + J^2(y)\right] Z_{\mathrm{stoc}}(\tau). \tag{3.3}$$

† For a recent introduction to stochastic differential equations see Schuss (1980).

‡ Note that this equation only defines Z up to a multiplicative constant, which is fixed by the normalisation condition $\langle 1\rangle_{\mathrm{ensemble}} = 1$.

4. Solution of generating functional equation

First consider the free field, with action given by

$$S=\frac{1}{2}\int \mathrm{d}x\,(-\partial^u\phi\,\partial_u\phi+m^2\phi^2). \tag{4.1}$$

The equation for the generating functional is

$$\frac{\partial Z_{\text{stoc}}}{\partial\tau}(\tau)=\int \mathrm{d}y\left(-J(y)(-\partial_y^2+m^2)\frac{\delta}{\delta J(y)}+J^2(y)\right)Z_{\text{stoc}}(\tau), \tag{4.2}$$

which can easily be solved by going to momentum space. Define

$$\tilde{J}(p)=\int \mathrm{d}y\,\mathrm{e}^{\mathrm{i}py}J(y), \tag{4.3}$$

then

$$\frac{\partial Z_{\text{stoc}}}{\partial\tau}(\tau)=\int \mathrm{d}p\left(-\tilde{J}(p)(p^2+m^2)\frac{\delta}{\delta\tilde{J}(p)}+\tilde{J}(p)\tilde{J}(-p)\right)Z_{\text{stoc}}(\tau), \tag{4.4}$$

with the solution (noting initial condition $Z_{\text{stoc}}(\tau=0)=1$)

$$Z_{\text{stoc}}(\tau)=\exp\left(\frac{1}{2}\int \mathrm{d}p\,\tilde{J}(p)\frac{1-\exp[-2\tau(p^2+m^2)]}{p^2+m^2}\tilde{J}(-p)\right). \tag{4.5}$$

This expression relaxes to the expected Feynman form for large τ, hence

$$\lim_{\tau\to\infty} Z_{\text{stoc}}(\tau)=Z_{\text{FT}}. \tag{4.6}$$

For the interacting field theory with a potential $V(\phi)$ (a polynomial in ϕ) and coupling constant λ the action is

$$S=\int \mathrm{d}x(-\tfrac{1}{2}\partial^u\phi\,\partial_u\phi+\tfrac{1}{2}m^2\phi^2+\lambda V(\phi)). \tag{4.7}$$

The generating functional obeys

$$\frac{\partial Z_{\text{stoc}}}{\partial\tau}(\tau)=\int \mathrm{d}y\left\{-J(y)\left[(-\partial_y^2+m^2)\frac{\delta}{\delta J(y)}+\lambda V'\left(\frac{\delta}{\delta J(y)}\right)\right]+J^2(y)\right\}Z_{\text{stoc}}(\tau), \tag{4.8}$$

where $'$ denotes differentiation.

This equation is difficult to solve for general τ. The stochastic quantisation hypothesis depends on equation (4.8) having a limiting solution for large τ or equivalently that

$$\lim_{\tau\to\infty}\frac{\partial Z_{\text{stoc}}}{\partial\tau}(\tau)=0. \tag{4.9}$$

This can be proved by perturbation theory in λ. Let

$$Z_{\text{stoc}}(\tau)=\sum_{n=0}^{\infty}\lambda^n Z_n(\tau), \tag{4.10}$$

and substitute in equation (4.8) to give

$$\frac{\partial Z_n}{\partial \tau} - \int dy\left[-J(y)\left((-\partial_y^2 + m^2)\frac{\delta}{\delta J(y)}\right) + J^2(y)\right] Z_n$$
$$= \int dy\, J(y)\, V'\left(\frac{\delta}{\delta J(y)}\right) Z_{n-1}, \qquad (4.11)$$

with $Z_{-1} = 0$.

Again it is best to go into momentum space. Introduce the free field Green functional $G_0[\tilde{J}, \tau, \tilde{J}', \tau']$ obeying

$$\left[\frac{\partial}{\partial \tau} - \int dp\left(-\tilde{J}(p)(p^2 + m^2)\frac{\delta}{\delta \tilde{J}(p)} + \tilde{J}(p)\tilde{J}(-p)\right)\right] G_0[\tilde{J}, \tau, \tilde{J}', \tau']$$
$$= \delta(\tau - \tau')\delta[\tilde{J} - \tilde{J}']. \qquad (4.12)$$

Equation (4.12) has the obvious solution

$$G_0[\tilde{J}, \tau, \tilde{J}', \tau'] = \theta(\tau - \tau')\delta\{\tilde{J}'(p) - \exp[-(\tau - \tau')(p^2 + m^2)\tilde{J}(p)]\}$$
$$\times \exp\left(\frac{1}{2}\int dp\, \tilde{J}(p)\frac{1 - \exp[-2(\tau - \tau')(p^2 + m^2)]}{p^2 + m^2}\tilde{J}(-p)\right). \qquad (4.13)$$

Using this Green functional the solution of equation (4.10) can be written

$$Z_n[\tilde{J}, \tau] = \int_0^\infty d\tau' \int D[\tilde{J}'(p)] G_0[\tilde{J}, \tau, \tilde{J}', \tau'] F_n[\tilde{J}', \tau'], \qquad (4.14)$$

where

$$F_n[\tilde{J}', \tau'] \equiv \int dy\, J'(y)\, V'\left(\frac{\delta}{\delta J'(y)}\right) Z_{n-1}[\tilde{J}', \tau'], \qquad (4.15)$$

is to be considered in momentum space. Making use of the form of the Green functional and setting $\tau - \tau' = s$ gives

$$Z_n[\tilde{J}, \tau] = \exp\left(\frac{1}{2}\int dp \frac{\tilde{J}(p)\tilde{J}(-p)}{p^2 + m^2}\right) \int_0^\tau ds\, F_n[\tilde{J} \exp(-s(p^2 + m^2), \tau - s]$$
$$\times \exp\left(-\frac{1}{2}\int dp \frac{\tilde{J}(p)\tilde{J}(-p) \exp[-2s(p^2 + m^2)]}{p^2 + m^2}\right). \qquad (4.16)$$

To verify equation (4.9) it is necessary to consider the structure of the F_n. For $n = 1$ it contributes two factors to the integral in equation (4.16):

$$\exp\left(\frac{1}{2}\int dp \frac{\tilde{J}(p)\tilde{J}(-p)}{p^2 + m^2} \exp[-2\tau(p^2 + m^2)]\right),$$

and a polynomial in $\tilde{J}(p)$, starting with the first power of $\tilde{J}(p)$ and each $\tilde{J}(p)$ multiplied by either $\exp[-s(p^2 + m^2)]$ or $\exp[-\tau(p^2 + m^2)]$. Hence the integrand in equation (4.16) is exponentially small for large τ and $\partial Z_1/\partial \tau \to 0$.

The above analysis can be repeated for $n = 2$ and then for $n = 3$ and so on for all n. Hence

$$\lim_{\tau \to \infty} \frac{\partial Z_n}{\partial \tau} = 0 \qquad \forall n. \qquad (4.17)$$

The large τ solution of equation (4.8) is now very simple: set $\partial Z/\partial\tau = 0$ and note the operator identity

$$\lambda V'\left(\frac{\delta}{\delta J(y)}\right) \exp\left[-\lambda \int \mathrm{d}x\, V\left(\frac{\delta}{\delta J(x)}\right)\right] = \left[J(y), \exp\left(-\lambda \int \mathrm{d}x\, V\left(\frac{\delta}{\delta J(x)}\right)\right)\right], \qquad (4.18)$$

which suggests the solution

$$Z_{\mathrm{stoc}}(\tau = \infty) = N \exp\left[-\lambda \int \mathrm{d}x\, V\left(\frac{\delta}{\delta J(x)}\right)\right] \exp\left(\frac{1}{2} \int \mathrm{d}p \frac{\tilde{J}(p)\tilde{J}(-p)}{p^2 + m^2}\right), \qquad (4.19)$$

where N is a normalisation constant. This proves the conjecture

$$\lim_{\tau\to\infty} Z_{\mathrm{stoc}}(\tau) = Z_{\mathrm{FT}}.$$

Indeed equation (4.19) is a good starting point for developing the Feynman perturbation series.

5. Outlook

Using techniques from stochastic calculus it has been shown directly that, for scalar fields, the stochastic quantisation conjecture holds. This should be compared with the more usual approach of introducing a τ dependent probability density obeying the Fokker-Planck equation and defining averages as functional integrals over the probability density.

This point might be particularly important for gauge theories where there are problems in defining functional integrals due to constraints between the fields. Constraints can be easily handled in the stochastic formulation as shown by Thomas (1984). Work is being actively carried out in this direction for gauge theories.

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